The Skin-Effect at High Frequencies

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(Invited Paper)

***Abstract*—During the last years, monolithic integrated circuits have been used more and more in microwave techniques. As a result, the metallization thickness of the planar circuits became of the order of the skin depth even at very high frequencies, so that the approximate methods for loss calculations used until recently must be revised. In this paper, a variational formulation of the skin-effect problem for calculating the losses as well as the inner inductances of components in such circuits will be described, and the first numerical results of the method will be discussed.**

I. INTRODUCTION

IN PLANAR MICROWAVE integrated circuits, the used line structures are striplines of rectangular cross section. The thickness of the metallization is between 5 and 20 μm in hybrid microwave integrated circuits and between 1 and 3 μm in monolithic microwave integrated circuits. Two different cases are interesting in real monolithic microwave integrated circuits

1) the lumped-element circuit, where the losses of a single conducting strip of rectangular cross section with no metallization on the backside of the substrate material must be calculated, and
2) the microstrip circuit, where the losses in the strip as well as in the ground plane must be calculated.

If it is assumed that the conductivity of the conductor material is in the order of $4 \cdot 10^7$ S/m, the skin depth of this material is about 3 μm at a frequency of 1 GHz and about 0.6 μm at a frequency of 20 GHz. Assuming that the height of the strips is between 1 and 3 μm, the current distribution is nearly constant over the cross section at the lower frequency (1 GHz), whereas, at 20 GHz, the skin-effect already has an important influence on the current distribution and thereby on the resistance of the strip.

This is the reason why in this paper the fundamental problem of the skin-effect in a conductor of rectangular cross section is considered again. Up to now, the loss calculations for high frequencies have been primarily based on the essential paper by Wheeler [1] in which an approximate method for calculating the frequency-dependent alternating current resistance of a conductor with dimensions large compared to the skin depth has been described (incremental inductance rule). In the case of thin-film lines as they are used in hybrid microwave integrated circuits, the requirements of Wheeler's theory are approximately fulfilled even at lower microwave frequencies because the metallization thickness of these circuits is between 5 and 10 μm. As has been shown above, in the case of monolithic microwave integrated circuits, and especially at lower frequencies, Wheeler's theory no longer can be applied because the skin depth is in the order of the metallization thickness.

In this first fundamental investigation, only the skin-effect in a single strip of rectangular cross section is considered. Using the results of this theory, the losses of lumped elements can be calculated directly. In addition, if the theory shall be applied to microstrip circuits, the magnetic-field distribution on the surface of the conducting areas or on a closed surface including the conducting structures must be known from other numerical methods. Investigations for this case will be published in the future.

The problem of the skin-effect in rectangular conductors is a very old one; one of the first papers dealing with the subject is that of Press (1916) [3]. In 1929, Cockcroft [4] published an approximate formula for the alternating current resistance of a rectangular conductor at high frequencies which he derived from an electrostatic analog of the problem. The requirements for the application of this formula are nearly the same as those for Wheeler's theory, i.e., both theories are applicable in a frequency range where the alternating current resistance is nearly proportional to the square-root of the frequency. In 1937, Haefner [5] published measured results for the frequency-dependent resistance of a rectangular conductor, which up to now are the only available measurements for resistances of different shape ratios and frequencies.

Already by 1927, Schwenkhagen [6] published an extensive investigation on a quasi-numerical method for the problem. This investigation was carried out for application in electrical power transmission, but it describes the same problems as those occurring in microwave integrated circuits today, especially the problem of the skin-effect resistance of coupled lines. During the following years, there had been a number of publications on numerical methods for calculating the alternating current resistance of a rectangular conductor [7]–[15]. The last publication was published in 1983, which showed that apparently the problem up until then had not been solved satisfactorily. In all papers, only few results of the numerical methods are presented; only Silvester [10] and Preis [14] give some detailed information on the frequency-dependent resistance; especially, all authors only give results for frequen-

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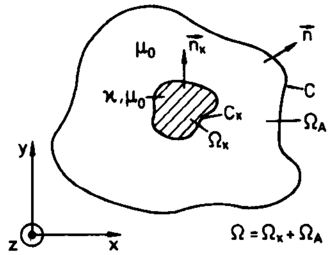


Fig. 1. The general cylindrical boundary-value problem.

cies which are relatively small, and the frequencies which are interesting for microwave applications (i.e., normalized frequencies p between $p = 2$ and $p = 30$; for the definition of p, see below) are not investigated. The highest value considered in the publications is $p = 6$. Additionally, little information is available in publications on the numerical effort, which in most cases is considered to be high.

In this paper, therefore, a numerical method based on a variational formulation of the problem will be described. In this first approach, only the single conducting strip of rectangular cross section without any substrate material and ground plane will be considered, i.e., the application of the method for calculating the losses of the microstrip lines will not be discussed. Already by 1978, Hammond [16], [17] remarked that the variational methods principally should be applicable to eddy current problems, and he verified this method for one-dimensional problems. Chun Hsiung Chen *et al.*, in 1980, published a general investigation on the application of variational methods in non-self-adjoint electromagnetic problems [18], and Sarkar [19], in 1984, gave interesting information on this subject and the inherent numerical difficulties. The author's paper is based on the referenced publications and tries to further develop the theoretical background and its application.

II. Variational Approach for the Skin-Effect

Because, in this paper, the investigation will be restricted to a single conducting strip of rectangular cross section, the quasi-stationary electromagnetic boundary-value problem shown in Fig. 1 is considered; it consists of a cylindrical arrangement with homogeneous cross section along the cylinder axis. The field domain Ω is subdivided into two sub-areas: Ω_κ is a domain described by a conductivity κ and a permeability $\mu = \mu_0$, whereas Ω_A is air-filled, i.e., the material parameter of this domain is $\mu = \mu_0$. All field regions are assumed to be homogeneous, i.e., all material parameters shall be constant in the sub-areas. The boundary curve C_κ surrounds the conducting area, and C is the boundary curve of the total field region Ω. Two different kinds of boundary conditions on the exterior curve C will be discussed later in this paper.

It is assumed that the electromagnetic field is independent of the z-coordinate (Fig. 1, cylinder axis); a current density in the conducting area shall exhibit only a z-component. Therefore, the adjoint magnetic field $\vec{B}$ is purely transversal and the electric field $\vec{E}$ as well as the vector potential $\vec{A}$ have only a z-component.

Under these conditions, Maxwell's equations describing the problem can be written as

$$\text{rot}\,\vec{B} = \mu\vec{S}_0 - j\omega\mu\kappa\vec{A}, \qquad \text{div}\vec{B} = 0 \tag{1}$$

$$\text{rot}\,\vec{A} = \vec{B}, \qquad \text{div}\vec{A} = 0 \tag{2}$$

where $\vec{S}_0$ is an impressed current density, which is assumed to be homogeneously distributed over the cross section of the area Ω_κ. $\vec{S}_0$ at the same time is the current density which will occur in the conducting area in the case $\omega \to 0$. The special form of (1) results from a Lorentz-gauge of the vector potential $\vec{A}$.

Using (1) and (2), it can easily be shown that the governing equation for the vector potential in the whole field domain $\Omega = \Omega_\kappa + \Omega_A$ is

$$\text{rotrot}\vec{A} + j\omega\mu\kappa\vec{A} = \mu\vec{S}_0 \qquad \text{in } \Omega. \tag{3}$$

This is the operator equation

$$L\vec{A} = \mu\vec{S}_0 \tag{4}$$

with the operator

$$L = \text{rotrot} + j\omega\mu\kappa \tag{5}$$

which will be used in the variational formulation of the problem later.

If the different field regions Ω_κ and Ω_A are considered separately, the following equations are valid for $\vec{A}$ and $\vec{B}$:

$$\left.\begin{aligned} \text{rotrot}\vec{A} + j\omega\mu\kappa\vec{A} = \mu\vec{S}_0 \\ \text{rotrot}\vec{B} + j\omega\mu\kappa\vec{B} = 0 \end{aligned}\right\} \qquad \text{in } \Omega_\kappa \tag{6}$$

and

$$\text{rotrot}\vec{A} = 0 \qquad \text{in } \Omega_A. \tag{7}$$

The solution of (4) or (6) and (7) needs the definition of the above-mentioned boundary condition on the curve C, which is the assumed outer boundary of the total field region. In a first case, a homogeneous Dirichlet boundary condition ($A_z = 0$), and in a second case, an inhomogeneous Neumann condition ($\partial A_z / \partial n = B_t(C)$) will be assumed. The first boundary condition $A_z = 0$ implies that the exterior curve C is a field-line of the magnetic field $\vec{B}$. The second boundary condition can be applied if a meaningful assumption on the value of the tangential magnetic flux-density $B_t(C)$ on the curve C can be made.

The operator L given in (5) is non-self-adjoint

$$\langle \vec{u}, L\vec{v} \rangle \neq \langle L\vec{u}, \vec{v} \rangle \tag{8}$$

where $\vec{u}$, $\vec{v}$ are vectors in the domain of the operator L. The scalar product in (8) is defined as

$$\langle \vec{u}, \vec{v} \rangle = \iint_\Omega \vec{u}^* \vec{v}\, d\Omega \tag{9}$$

where Ω is the considered field region.

As it is known from [18] in the case of the non-self-adjoint problem given in (4), an adjoint operator L^a can be defined so that

$$\langle \vec{u}, L\vec{v} \rangle = \langle L^a \vec{u}, \vec{v} \rangle. \tag{10}$$

The form of the adjoint operator L^a and the general

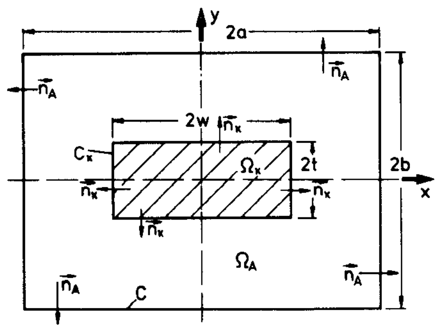


Fig. 2. The conducting cylinder of rectangular cross section (region Ω_κ, boundary C_κ) and the exterior boundary C.

procedure of solving the skin-effect problem of the rectangular conductor shall be explained for a special example in the next chapter.

III. The Skin-Effect of a Conductor with Rectangular Cross Section

A. The Variational Method Using One Vector Potential

In this section, the problem in Fig. 2 shall be analyzed using the described method. On the exterior boundary C, the homogeneous Dirichlet condition ($A_z = 0$) shall be assumed. The operator equation (4) (with the operator given in (5)), which shall be solved, is valid in the total field region Ω surrounded by C. The adjoint operator L^a can be derived in this case as

$$L^a = L^* = \mathrm{rotrot} - j\omega\mu\kappa \tag{11}$$

where L^* means the conjugate complex of L (see also [17]). Additionally, the adjoint vector potential $\vec{A}^a$, which fulfills

$$L^a \vec{A}^a = \mu \vec{S}_0 \tag{12}$$

(where the impressed adjoint current density $\vec{S}_0^a$ can be chosen to be $\vec{S}_0^a = \vec{S}_0$ without any restrictions) must satisfy the same homogeneous boundary conditions as $\vec{A}$, so that the "surface integrals" appearing when Green's formula is applied to $\langle \vec{u}, L\vec{v} \rangle$, and using (11)

$$\langle \vec{u}, L\vec{v} \rangle = \langle L^a \vec{u}, \vec{v} \rangle + [\vec{v} \times \mathrm{rot}\,\vec{u}^*, \vec{n}] - [\vec{u}^* \times \mathrm{rot}\,\vec{v}, \vec{n}] \tag{13}$$

automatically become zero. The surface integrals in (13) are defined as

$$[\vec{v} \times \mathrm{rot}\,\vec{u}^*, \vec{n}] = \oint_C (\vec{v} \times \mathrm{rot}\,\vec{u}^*) \cdot \vec{n}\, ds. \tag{14}$$

In addition, the surface integrals must be taken into account if not the essential boundary condition $A_z = 0$, but the natural conditions, e.g., the inhomogeneous Neumann condition, are considered.

Following [20] and [18] in the case of the homogeneous boundary condition, a solution for the field vectors $\vec{A}$ and $\vec{A}^a$ can be derived from the functional

$$F(\vec{A}, \vec{A}^a) = \langle \vec{A}^a, L\vec{A} \rangle - \langle \mu \vec{S}_0, \vec{A} \rangle - \langle \vec{A}^a, \mu \vec{S}_0 \rangle \tag{15}$$

which must be stationary in the sense that

$$\delta F(\vec{A}, \vec{A}^a) = 0. \tag{16}$$

To find a solution for the vector potentials $\vec{A}$ and $\vec{A}^a$, use is made of a series expansion

$$A_z = \sum_{i=1}^{\infty} \alpha_i \varphi_i(x, y)$$
$$A_z^a = \sum_{j=1}^{\infty} \alpha_j^a \varphi_j^a(x, y) \tag{17}$$

where the expansion functions $\{\varphi_i\}$ and $\{\varphi_j^a\}$ should be linearly independent and form a complete set of functions. Additionally, the expansion functions must fulfill separately the adjoint essential boundary conditions. The coefficients α_i and α_j^a can be determined by carrying out the first variation of (16)

$$\frac{\partial F}{\partial \alpha_k^a} = \sum_{i=1}^{\infty} \alpha_i \langle \varphi_k^a \vec{e}_z, L(\varphi_i \vec{e}_z) \rangle - \langle \varphi_k^a \vec{e}_z, \mu \vec{S}_0 \rangle = 0 \tag{18a}$$

$$\frac{\partial F}{\partial \alpha_l} = \sum_{j=1}^{\infty} \alpha_j^a \langle \varphi_l \vec{e}_z, L^a(\varphi_j^a \vec{e}_z) \rangle - \langle \varphi_l \vec{e}_z, \mu \vec{S}_0 \rangle = 0,$$

$$\text{for all } k, l = 1, 2, 3, \cdots. \tag{18b}$$

As can be seen, the determination of the coefficients α_i, α_j^a from (18) is identical to the requirement that

$$\langle \varphi_k^a \vec{e}_z, L(A_z \vec{e}_z) - \mu \vec{S}_0 \rangle = 0 \tag{19a}$$

$$\langle \varphi_l \vec{e}_z, L^a(A_z^a \vec{e}_z) - \mu \vec{S}_0 \rangle = 0. \tag{19b}$$

This means that the coefficients α_i are chosen in such a way that $L(A_z \vec{e}_z) - \mu \vec{S}_0$ is orthogonal to the expansion functions $\varphi_k^a \vec{e}_z$ and the coefficients α_j^a are chosen so that $L^a(A_z^a \vec{e}_z) - \mu \vec{S}_0$ is orthogonal to $\varphi_l \vec{e}_z$. Sarkar [19] has shown that this property can lead to numerical difficulties in the solution process.

In the special case considered here, the boundary conditions for the original and the adjoint problem ($A_z = 0$, $A_z^a = 0$) are identical; therefore, it is possible to make use of Galerkin's method, i.e., $\varphi_i = \varphi_i^a$.

It is convenient for the series expansion to separate the vector potentials into their real and imaginary parts, respectively

$$A_z = \sum_{i=1}^{\infty} \beta_i \varphi_i'(x, y) + j \sum_{i=1}^{\infty} \gamma_i \varphi_i''(x, y)$$
$$A_z^a = \sum_{j=1}^{\infty} \beta_j^a \varphi_j'(x, y) + j \sum_{j=1}^{\infty} \gamma_j^a \varphi_j''(x, y). \tag{20}$$

Without any restrictions, the real and imaginary parts of the expansion functions can be chosen to be equal. Introducing (20) into (18) under the mentioned assumptions, after some lengthy mathematics, it can be shown that the resulting equation systems can be written in the form

$$\langle \mathrm{rot}\vec{A}', \mathrm{rot}\,\delta\vec{A}' \rangle_\Omega - \omega\mu\kappa \langle \vec{A}'', \delta\vec{A}' \rangle_{\Omega_\kappa} = \mu \langle \vec{S}_0, \delta\vec{A}' \rangle_{\Omega_\kappa}$$
$$\langle \mathrm{rot}\vec{A}'', \mathrm{rot}\,\delta\vec{A}' \rangle_\Omega + \omega\mu\kappa \langle \vec{A}', \delta\vec{A}' \rangle_{\Omega_\kappa} = 0 \tag{21}$$

and

$$\langle \mathrm{rot}\vec{A}', \mathrm{rot}\,\delta\vec{A}'' \rangle_\Omega - \omega\mu\kappa \langle \vec{A}'', \delta\vec{A}'' \rangle_{\Omega_\kappa} = \mu \langle \vec{S}_0, \delta\vec{A}'' \rangle_{\Omega_\kappa}$$
$$\langle \mathrm{rot}\vec{A}'', \mathrm{rot}\,\delta\vec{A}'' \rangle_\Omega + \omega\mu\kappa \langle \vec{A}', \delta\vec{A}'' \rangle_{\Omega_\kappa} = 0 \tag{22}$$

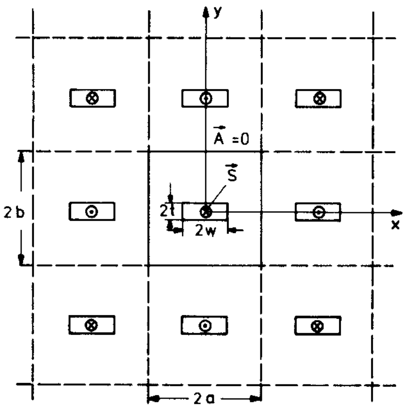


Fig. 3. The physical arrangement of an infinite number of rectangular conductors and their currents which must be considered to verify the boundary condition $A_z = 0$ on C.

which means that (18a) and (18b) lead to one and the same equation system for the vector potential $\vec{A}$ according to (20).

If the coefficients β_i and γ_i are determined, the current density $\vec{S}$ can be found from

$$\vec{S} = \vec{S}_0 - j\omega\kappa\vec{A} \tag{23}$$

where $(-j\omega\kappa\vec{A})$ is the eddy current density which is superimposed to the homogeneous distributed current density $\vec{S}_0$. The alternating current resistance R normalized to the dc current resistance R_0 is given by

$$\frac{R}{R_0} = 4wt \frac{\iint_{\Omega_\kappa} |\vec{S}_0 - j\omega\kappa\vec{A}|^2 \, d\Omega}{\left| \iint_{\Omega_\kappa} \vec{S}_0 - j\omega\kappa\vec{A} \, d\Omega \right|^2}. \tag{24}$$

Some additional aspects of the boundary condition $A_z = 0$ as it was assumed above shall be given shortly. As it was pointed out before, the condition $A_z = 0$ can be interpreted in the way that the boundary curve C is a magnet flux line. Physically, this can be realized by an infinite arrangement of rectangular conductors with positive and negative current densities as shown in Fig. 3. Using this method of images, the calculation of eddy current problems in coupled line structures can also be carried out. Depending on the orientation and the position of the image currents, different boundary conditions can be treated; so, for example, the additional influence of the proximity effect can also be calculated.

In some problems, it is possible to make a meaningful assumption on the distribution of the tangential magnetic field in the exterior boundary C. In this case, the inhomogeneous Neumann boundary conditions apply, and the final system equation for calculating the expansion coefficients can be written as

$$\langle \mathrm{rot}\vec{A}', \mathrm{rot}\delta\vec{A}' \rangle_\Omega - \omega\mu\kappa \langle \vec{A}'', \delta\vec{A}' \rangle_{\Omega_\kappa} = \langle \mu\vec{S}_0, \delta\vec{A}' \rangle_{\Omega_\kappa} + \left[\delta\vec{A}' \times \vec{B}'(C), \vec{n} \right]_C$$

$$\langle \mathrm{rot}\vec{A}'', \mathrm{rot}\delta\vec{A}' \rangle_\Omega + \omega\mu\kappa \langle \vec{A}', \delta\vec{A}' \rangle_{\Omega_\kappa} = 0 \tag{25}$$

where $\vec{B} = \vec{B}'$ is assumed to be real on the boundary C. In contrast to the case of the essential boundary conditions $(A_z = 0)$, the expansion functions are unrestricted on the exterior boundary C here.

B. *The Variational Method Using Two Vector Potentials*

As shown in Fig. 1, the cross section of the cylindrical field problem can be subdivided into a conducting and a nonconducting field region. In Section III-A, only one series expansion was used for describing the vector potential in both field regions. If the ratio Ω_κ/Ω_A is small, this method may be disadvantageous because the number of expansion coefficients which are needed to find an approximate solution of the infinite system equation (e.g., (21) and (22)) becomes large. Alternatively, two different series expansions can be used for the vector potential in the areas Ω_κ and Ω_A. This method leads to a system equation similar to (25), where the surface integral over the inner boundary C_κ must be used to fulfill the continuity condition of the magnetic field in this boundary

$$\langle \mathrm{rot}\vec{A}'_\kappa, \mathrm{rot}\delta\vec{A}'_\kappa \rangle_{\Omega_\kappa} - \omega\mu\kappa \langle \vec{A}''_\kappa, \delta\vec{A}'_\kappa \rangle_{\Omega_\kappa} - \left[\delta\vec{A}'_\kappa \times \mathrm{rot}\vec{A}'_A, \vec{n}_\kappa \right]_{C_\kappa} = \langle \mu\vec{S}_0, \delta\vec{A}'_\kappa \rangle$$

$$\langle \mathrm{rot}\vec{A}''_\kappa, \mathrm{rot}\delta\vec{A}'_\kappa \rangle_{\Omega_\kappa} + \omega\mu\kappa \langle \vec{A}'_\kappa, \delta\vec{A}'_\kappa \rangle_{\Omega_\kappa} - \left[\delta\vec{A}'_\kappa \times \mathrm{rot}\vec{A}''_A, \vec{n}_\kappa \right]_{C_\kappa} = 0 \tag{26}$$

and

$$\left[\delta\vec{A}'_A \times \mathrm{rot}\,\vec{A}'_\kappa, \vec{n}_\kappa \right]_{C_\kappa} + \langle \mathrm{rot}\,\vec{A}'_A, \mathrm{rot}\,\delta\vec{A}'_A \rangle_{\Omega_A} = \left[\delta\vec{A}'_A \times \vec{B}'(C), \vec{n} \right]_C$$

$$\left[\delta\vec{A}'_A \times \mathrm{rot}\,\vec{A}''_\kappa, \vec{n}_\kappa \right]_{C_\kappa} + \langle \mathrm{rot}\,\vec{A}''_A, \mathrm{rot}\,\delta\vec{A}'_A \rangle_{\Omega_A} = 0 \tag{27}$$

where $\vec{A}_\kappa$ is the vector potential in the conducting area Ω_κ and $\vec{A}_A$ is the vector potential in Ω_A. For both vector potentials, series expansions equivalent to (20), and considering the requirements of Galerkin's method are defined, the coupling integrals over the interface C_κ guarantee that the continuity condition for the tangential magnetic field is fulfilled so that $\partial\vec{A}/\partial n$ is continuous. Additionally, the essential continuity condition for $\vec{A}$ itself must be fulfilled explicitly by a proper choice of the series expansion. In (27), both kinds of the boundary conditions on the exterior curve C are considered simultaneously. In the case of the essential boundary condition $(A_z = 0)$, $\delta\vec{A}'_A$ is zero on C; therefore, the surface integral vanishes, whereas in the case of the natural boundary condition, the surface integral introduces the prescribed value of $\vec{B}_{\tan}$ on C into the equation system.

C. *The Variational Method Using the Magnetic Flux and the Magnetic Vector Potential*

Using the method described in Section III-B, it may sometimes become difficult to fulfill the essential continuity condition for $\vec{A}$ explicitly. In this case, a third method, which shall be described below, may be helpful. Instead of using the magnetic vector potential in the area Ω_κ as in the area Ω_A, the magnetic flux $\vec{B}$ is used to describe the fields in Ω_κ; the fields in Ω_A again are derived from the vector potential. The field equations in the two areas therefore are

$$\operatorname{rotrot} \vec{B} + j\omega\mu\kappa\vec{B} = 0 \quad \text{in } \Omega_\kappa$$
$$\operatorname{rotrot} \vec{A} = 0 \quad \text{in } \Omega_A. \tag{28}$$

Using the way of derivation which principally has been explained in Section III-A, the equation systems in Ω_κ and Ω_A take the form

$$\langle \operatorname{rot} \vec{B}', \operatorname{rot} \delta\vec{B}'\rangle_{\Omega_\kappa} - \omega\mu\kappa\langle \vec{B}'', \delta\vec{B}'\rangle_{\Omega_\kappa} - \left[\delta\vec{B}' \times \operatorname{rot} \vec{B}', \vec{n}_\kappa\right]_{C_\kappa} = 0$$
$$\langle \operatorname{rot} \vec{B}'', \operatorname{rot} \delta\vec{B}'\rangle_{\Omega_\kappa} + \omega\mu\kappa\langle \vec{B}', \delta\vec{B}'\rangle_{\Omega_\kappa} - \left[\delta\vec{B}' \times \operatorname{rot} \vec{B}'', \vec{n}_\kappa\right]_{C_\kappa} = 0 \tag{29}$$

in Ω_κ and on C_κ and

$$\langle \operatorname{rot} \vec{A}', \operatorname{rot} \delta\vec{A}'\rangle_{\Omega_A} + \left[\delta\vec{A}' \times \operatorname{rot} \vec{A}', \vec{n}_\kappa\right]_{C_\kappa} - \left[\delta\vec{A}' \times \operatorname{rot} \vec{A}', \vec{n}\right]_C = 0$$
$$\langle \operatorname{rot} \vec{A}'', \operatorname{rot} \delta\vec{A}'\rangle_{\Omega_A} + \left[\delta\vec{A}' \times \operatorname{rot} \vec{A}'', \vec{n}_\kappa\right]_{C_\kappa} - \left[\delta\vec{A}' \times \operatorname{rot} \vec{A}'', \vec{n}\right]_C = 0 \tag{30}$$

in Ω_A and on C_κ as well as on C.

Because, in (29) and (30), the set up for the magnetic flux density in Ω_κ and the magnetic vector potential in Ω_A are still independent of each other, the continuity condition on C_κ must be used to determine the necessary coupling between these two fields. Using the continuity of the tangential magnetic-field strength and the continuity of the vector potential, the following equation system can be derived:

$$\langle \operatorname{rot} \vec{B}', \operatorname{rot} \delta\vec{B}'\rangle_{\Omega_\kappa} - \omega\mu\kappa\langle \vec{B}'', \delta\vec{B}'\rangle_{\Omega_\kappa} - \left[\delta\vec{B}' \times \omega\mu\kappa\vec{A}'', \vec{n}_\kappa\right]_{C_\kappa} = \left[\delta\vec{B}' \times \mu\vec{S}_0, \vec{n}_\kappa\right]_{C_\kappa}$$
$$\langle \operatorname{rot} \vec{B}'', \operatorname{rot} \delta\vec{B}'\rangle_{\Omega_\kappa} + \omega\mu\kappa\langle \vec{B}', \delta\vec{B}'\rangle_{\Omega_\kappa} + \left[\delta\vec{B}' \times \omega\mu\kappa\vec{A}', \vec{n}_\kappa\right]_{C_\kappa} = 0 \tag{31}$$

in Ω_κ and on C_κ, and

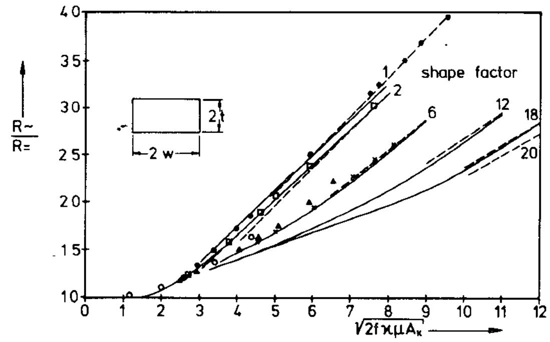
$$\langle \operatorname{rot} \vec{A}', \operatorname{rot} \delta\vec{A}'\rangle_{\Omega_A} + \left[\delta\vec{A}' \times \vec{B}', \vec{n}_\kappa\right]_{C_\kappa} = \left[\delta\vec{A}' \times \vec{B}'(C), \vec{n}\right]_C$$
$$\langle \operatorname{rot} \vec{A}'', \operatorname{rot} \delta\vec{A}'\rangle_{\Omega_A} + \left[\delta\vec{A}' \times \vec{B}'', \vec{n}_\kappa\right]_{C_\kappa} = 0 \tag{32}$$

in Ω_A as well as on C_κ and C.

Depending on the applied boundary condition, the surface integral over C vanishes ($A_z = 0$) or introduces the value of the tangential magnetic flux on C into the problem.

IV. First Numerical Results and Discussion of the Method

First numerical results for the problem of the rectangular conductor shown in Fig. 2 are available and shall be discussed here. In all computations, the essential boundary condition ($A_z = 0$) as it has been described above has been used.. One part of the results presented here has been derived using the method described in Section III-A; for the shape factor $w/t = 6$ (Fig. 4) the method in Section III-B has been used additionally; the results of the two methods will be compared. Depending on the structure of the conductor and the surrounding area, the application of one or the other method may give a higher flexibility in the formulation of the expansion functions and may thereby be advantageous.



w / µm	t / µm	p at 1 GHz	p at 10 GHz	p at 20 GHz
1	1	0657	2079	294
2	2	1314	415	588
5	2	2079	6574	9297
5	5	3287	10789	147
20	5	6574	20789	294

Fig. 4. The measured alternating current resistance R normalized to the dc resistance R_0 ([5], solid lines) in dependence on the normalized frequency $p = \sqrt{2f\kappa\mu A_\kappa}$ and numerical results of the method described in Section III-A. (Shape ratio SR:● $SR = 1$, □ $SR = 2$, ○ $SR = 4$, △ $SR = 6$) and in Section III-B (Shape ratio SR: × $SR = 6$)

Using the method described in Section III-A, the vector potential $\vec{A}$ is expanded into a series of trigonometric functions

$$A'_z = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} a_{ij} \cos\left[(2i-1)\frac{\pi x}{2a}\right] \cos\left[(2j-1)\frac{\pi y}{2b}\right]$$
$$A''_z = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} b_{ij} \cos\left[(2i-1)\frac{\pi x}{2a}\right] \cos\left[(2j-1)\frac{\pi y}{2b}\right] \tag{33}$$

where N_x and N_y are the highest values which are considered in the numerical evaluation. The expansion functions used in (33) form a complete set as N_x, N_y approach infinity (entire basis functions), they are orthogonal in Ω and each element of the function system is zero on the boundary C as it is required by the essential boundary condition.

Formulating the equation system (22) with the expansion function (33), a $N \times N$ system matrix results where $N = 2N_xN_y$. Using the orthogonality of the expansion functions, it can be shown that the matrix can be split up into two frequency-dependent diagonal submatrices and two frequency-independent full submatrices, so that the equation system can be rearranged reducing the matrix size by a factor 4. Because of the special form of the system matrix, only the main diagonal elements of the resulting matrix change with the frequency, so that in the calculations at different frequencies only few elements of the matrix must be recalculated.

For the special problem shown in Fig. 2, up to 20 elements (N_x, N_y) of the series expansions have been used corresponding to a 400×400 matrix. The resulting CPU time for solving the problem at one frequency on a CDC 7600 is about 60 s; after one computation in which the matrix is generated once, the CPU time reduces to about 10 s for each further frequency point. The 20 elements of the series expansions are needed because the exterior curve C must be sufficiently far away from the inner boundary C_κ to avoid an influence of the exterior boundary C on the current distribution inside the conductor. Best results were obtained using a quadratic exterior boundary C with $a=b$, where a and b are at least three times larger than the larger value of w or t.

In Fig. 4, the measured alternating current resistances published by Haefner [5] are shown as a function of the normalized frequency p

$$p=\sqrt{2f\kappa\mu(4\omega t)}=\sqrt{2f\kappa\mu A_\kappa} \tag{34}$$

where A_κ is the cross-section area of the conductor. The use of the normalized frequency is helpful in so far as it allows one to compare the resistances of conductors with the same shape ratio as it is required by the principle of similitude. As can be seen from the table in Fig. 4, the p-values interesting for application in microwave monolithic integrated circuits are between 0.6 and about 30. For the high values ($p\geqq 8$–10), Wheeler's [1] or Cockcroft's [4] theory are quite good approximations, as can be seen from Fig. 4 (dashed curves). Especially for high values of the shape ratio and small values of p, the differences between the approximate theories and the measurements become large because these approximations describe the alternating current resistance as linearly dependent on the normalized frequency p; therefore, the dashed straight lines are always beginning in the origin of the coordinate system (i.e., $R=0$ for $p=0$).

Fig. 4 also shows the first numerical results computed by the described method. For all shape ratios and small values of the normalized frequency p ($p\leqq5$), the agreement between the theoretically computed values and Haefner's experimental data is quite good. The same is true for shape ratios between 1 and 2 (i.e., nearly quadratic conductors) up to frequencies $p=10$ (which is the highest value for which computations have been done up to now). Because the quadratic conductor is the one in which the two-dimensional current displacement is most marked, the good agreement in this case is considered a confirmation of the theoretical method. The disagreement between Haefner's measurements and the computed results for the shape ratio $w/t=6$ and for higher p-values may be a result of insufficiently high cuts in the series expansions; higher values than $N_x=N_y=20$ could not yet be achieved on the formerly available computer.

Additionally, the method described in Section III-B has been applied; the results for the shape factors $w/t=1$ and 2 do not differ from those computed with the method in Section III-A. In the case of the shape factor $w/t=6$, the method in Section III-B gives a better possibility to compute the resistances as can be expected by the small ratio Ω_κ/Ω_A. As can be seen from Fig. 4, the results calculated with the method in Section III-B agree well with Haefner's measurement, and for high frequencies (high p-values), Cockcroft's approximation is verified.

V. Summary

As a result, it can be pointed out that for the first time the measurements published by Haefner [5] for the alternating current resistance of the rectangular bar have been verified by a numerical calculation method for the skin-effect in a large frequency range. For very high frequencies, it is shown that Cockcroft's approximate formulas can be applied.

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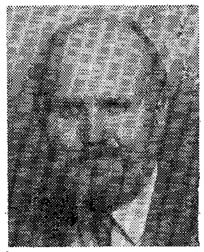

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